

EXISTENCE FOR STATIONARY MEAN FIELD GAMES WITH QUADRATIC HAMILTONIANS WITH CONGESTION

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ABSTRACT. In this paper, we investigate the existence and uniqueness of solutions to a stationary mean field game model introduced by J.-M. Lasry and P.-L. Lions. This model features a quadratic Hamiltonian with possibly singular congestion effects. Thanks to a new class of a-priori bounds, combined with the continuation method, we prove the existence of smooth solutions in arbitrary dimensions.

1. INTRODUCTION

Since the seminal papers [HMC06, HCM07, LL06a, LL06b, LL07a, LL07b], research on mean field games has been extremely active (see, for instance, the recent surveys [LLG10, Car11, Ach13, GS14] and the references therein). Nevertheless, several fundamental questions have not yet been answered. In this paper, we address one of those, and prove existence and uniqueness of smooth solutions for stationary mean field games with congestion and quadratic Hamiltonian.

Mean field games model large populations of rational agents who move according to certain stochastic optimal control goals. To simplify the presentation, we will work in the periodic setting, that is in the d dimensional standard torus \mathbb{T}^d , $d \geq 1$. We consider a large stationary population of agents, whose statistical information is encoded in an unknown probability density $m : \mathbb{T}^d \times [0, +\infty) \rightarrow \mathbb{R}$. Each individual agent wants to minimize an infinite horizon discounted cost given by

$$u(x, t) = \inf \left\{ \mathbb{E} \left[\int_t^{+\infty} e^{-s} \left(\frac{m(X(s), s)^\alpha |v(s) - b(X(s))|^2}{2} + V(X(s), m(X(s), s)) \right) ds \right] \right\},$$

where the infimum is taken over all progressively measurable controls v ,

$$dX = vdt + \sqrt{2}dW_t \text{ with } X(0) = x,$$

W_t and \mathbb{E} denote a standard d -dimensional Brownian motion and the expected value, respectively. The constant $0 < \alpha < 1$ determines the strength of congestion effects in the term $m^\alpha |v - b(x)|^2$, and makes it costly to move in areas of high density with a drift v substantially different from a reference vector field $b : \mathbb{T}^d \rightarrow \mathbb{R}^d$. The function $V : \mathbb{T}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ accounts for additional spatial preferences of the agents. We assume b and V to be smooth functions.

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Under standard assumptions of rationality and symmetry, one can derive a mean field problem which models this setup. A detailed discussion can be found in [Lio11, LL07a], where the following problem, consisting of a viscous Hamilton-Jacobi equation for u , coupled with a Fokker-Planck equation for m , was introduced:

$$\begin{cases} -u_t + u - \Delta u + \frac{|Du|^2}{2m^\alpha} + b(x) \cdot Du = V(x, m) \\ m_t + m - \Delta m - \operatorname{div}(m^{1-\alpha} Du) - \operatorname{div}(mb) = 0, \end{cases}$$

together with initial conditions for $m(x, 0)$ and suitable asymptotic behavior for u .

In the present paper, we consider a stationary version of this problem, which for $u, m : \mathbb{T}^d \rightarrow \mathbb{R}$, $m > 0$, is given by the system

$$\begin{cases} u - \Delta u + \frac{|Du|^2}{2m^\alpha} + b(x) \cdot Du = V(x, m) & \text{in } \mathbb{T}^d, \end{cases} \quad (1.1)$$

$$\begin{cases} m - \Delta m - \operatorname{div}(m^{1-\alpha} Du) - \operatorname{div}(mb) = 1, & \text{in } \mathbb{T}^d. \end{cases} \quad (1.2)$$

where the right hand side of the second equation is an additional source term for m (to avoid the trivial solution $m = 0$). In [Lio11], only the uniqueness of smooth solutions was proven. However, existence of solutions, the main result of this paper, was not yet known for both stationary and time-dependent problems. The fundamental difficulty lies in the possibly singular behavior due to congestion. The dependence on m in the optimal control problem causes the singularity in the equation (1.1), for which we had to develop a new class of estimates. Thanks to those we obtain our main result, which is

Theorem 1.1. *Assume the following:*

- (A1) $0 \leq \alpha < 1$;
- (A2) $V : \mathbb{T}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $V(x, m) \in C^\infty(\mathbb{T}^d \times \mathbb{R}^+)$ is globally bounded with bounded derivatives and non-decreasing with respect to m ;
- (A3) $b : \mathbb{T}^d \rightarrow \mathbb{R}^d$, $b \in C^\infty(\mathbb{T}^d)$.

Then there exists a solution $(u, m) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$ to (1.1)-(1.2) with $m > 0$. Furthermore, if V is strictly increasing with respect to m , then a solution is unique.

Numerous a-priori bounds for mean field games have been proved by various authors, including the first author (see, for instance [LL06a], [LL06b], [LL07a], [GSM14], [GISMY10], [CLLP12], [GPSM12], [GPSM14], [GPSM13], [GPV14], [Por14], [Por13]). However, these bounds were designed to address a different coupling, namely mean field games where the local dependence on m is not singular when $m = 0$. A typical example is the following system

$$\begin{cases} u - \Delta u + \frac{|Du|^2}{2} = m^\alpha & \text{in } \mathbb{T}^d \\ m - \Delta m - \operatorname{div}(m Du) = 1 & \text{in } \mathbb{T}^d. \end{cases} \quad (1.3)$$

In (1.3), the main difficulties are caused by the growth of the nonlinearity m , especially for large $\alpha > 0$, rather than singularities caused by m vanishing. Besides, (1.3) can be regarded as an Euler-Lagrange equation of a suitable functional, whereas (1.1)-(1.2) does not have this structure.

In Section 2, we start by exploring the special form of (1.1) and (1.2) to obtain a bound for $\|m^{-1}\|_{L^\infty(\mathbb{T}^n)}$. This estimate, combined with the techniques from [AC78], yields

a-priori regularity in $W^{2,p}(\mathbb{T}^d)$ for any $p \geq 1$. From this, a simple argument shows that any solution to (1.1)-(1.2) is bounded in any Sobolev space $W^{k,p}(\mathbb{T}^d)$. Then, in Section 3, we prove the existence of solutions to (1.1)-(1.2) by using the continuation method together with the aforementioned a-priori estimates. In Appendix A, for completeness, we present the uniqueness proof for solutions to (1.1)-(1.2), based upon the ideas in [Lio11] (see also [Gue14]). In a forthcoming paper [GM13], we will study general mean field games with congestion, for which the techniques of the present paper cannot be applied directly, as discussed in Remark 1 at the end of the next section.

2. A-PRIORI ESTIMATES

In this section, we obtain a-priori bounds for solutions of (1.1)-(1.2). In particular, we prove a L^∞ bound for m^{-1} . From this, we derive estimates for u, m in $W^{2,p}(\mathbb{T}^d)$ for any $p \geq 1$. Then, by a bootstrapping argument, we establish smoothness of solutions.

Proposition 2.1. *There exists a constant $C := C(\|V\|_\infty) \geq 0$ such that for any classical solution (u, m) of (1.1)-(1.2) we have $\|u\|_{L^\infty(\mathbb{T}^d)} \leq C$. Furthermore, $m \geq 0$ on \mathbb{T}^d , and $\|m\|_{L^1(\mathbb{T}^d)} = 1$.*

Proof. The L^∞ bound is obtained by evaluating the equation at points of maximum of u (resp., minimum) and using the facts that at those points $Du = 0$, $\Delta u \leq 0$ (resp., ≥ 0), and V is bounded on $\mathbb{T}^d \times [0, \infty)$. We then observe that m is non-negative by the maximum principle. Moreover, it has total mass 1 by integrating (1.2). \square

Proposition 2.2. *There exists a constant $C := C(\|b\|_\infty, \|V\|_\infty) \geq 0$ such that for any classical solution (u, m) of (1.1)-(1.2) we have*

$$\left\| \frac{1}{m} \right\|_{L^\infty(\mathbb{T}^d)} \leq C.$$

Proof. Let $r > \alpha$. Subtract equation (1.2) divided by $(r+1-\alpha)m^{r+1-\alpha}$ from equation (1.1) divided by rm^r . Then,

$$\begin{aligned} & \int_{\mathbb{T}^d} \left[u - \Delta u + \frac{|Du|^2}{2m^\alpha} + b \cdot Du - V \right] \cdot \frac{1}{rm^r} dx \\ & - \int_{\mathbb{T}^d} \left[m - \Delta m - \operatorname{div}(m^{1-\alpha} Du) - \operatorname{div}(mb) \right] \cdot \frac{1}{(r+1-\alpha)m^{r+1-\alpha}} dx \\ & = - \int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}} dx. \end{aligned} \tag{2.1}$$

Next, observe that

$$\int_{\mathbb{T}^d} \frac{\Delta u}{rm^r} dx = \int_{\mathbb{T}^d} \frac{Du \cdot Dm}{m^{r+1}} dx,$$

and

$$\int_{\mathbb{T}^d} \frac{\operatorname{div}(m^{1-\alpha} Du)}{(r+1-\alpha)m^{r+1-\alpha}} dx = \int_{\mathbb{T}^d} \frac{Du \cdot Dm}{m^{r+1}} dx.$$

Hence

$$\int_{\mathbb{T}^d} \frac{\Delta u}{rm^r} dx - \int_{\mathbb{T}^d} \frac{\operatorname{div}(m^{1-\alpha} Du)}{(r+1-\alpha)m^{r+1-\alpha}} dx = 0. \tag{2.2}$$

Also, note the identity

$$\begin{aligned} \int \operatorname{div}(bm) \frac{m^{-r-1+\alpha}}{r+1-\alpha} &= \int m^{-r-1+\alpha} b \cdot Dm \\ &= - \int b \cdot D \left(\frac{m^{-r+\alpha}}{r-\alpha} \right) = \frac{1}{r-\alpha} \int \operatorname{div}(b) m^{-r+\alpha}. \end{aligned}$$

Then (2.1) is reduced to

$$\begin{aligned} & \int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}} dx + \int_{\mathbb{T}^d} \frac{|Du|^2}{2rm^{r+\alpha}} dx + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}} dx \\ &= \int_{\mathbb{T}^d} \left[-\frac{V}{rm^r} - \frac{u}{rm^r} + \frac{1}{(r+1-\alpha)m^{r-\alpha}} - \frac{b \cdot Du}{rm^r} - \frac{1}{r-\alpha} \operatorname{div}(b) m^{-r+\alpha} \right] dx \\ &\leq \int_{\mathbb{T}^d} \frac{C}{rm^r} dx + \int_{\mathbb{T}^d} \frac{C}{(r-\alpha)m^{r-\alpha}} dx + \int_{\mathbb{T}^d} \frac{|b|^2}{rm^{r-\alpha}} + \frac{|Du|^2}{4rm^{r+\alpha}} dx \end{aligned}$$

in view of Proposition 2.1. Consequently,

$$\begin{aligned} & \int_{\mathbb{T}^d} \frac{1}{(r+1-\alpha)m^{r+1-\alpha}} dx + \int_{\mathbb{T}^d} \frac{|Du|^2}{4rm^{r+\alpha}} dx + \int_{\mathbb{T}^d} \frac{|Dm|^2}{m^{r+2-\alpha}} dx \\ &\leq \int_{\mathbb{T}^d} \frac{C}{rm^r} dx + \int_{\mathbb{T}^d} \frac{C}{(r-\alpha)m^{r-\alpha}} dx. \end{aligned}$$

By Young's inequality, for $\alpha \in [0, 1)$, we have

$$\frac{C}{rm^r} \leq \frac{1}{4(r+1-\alpha)m^{r+1-\alpha}} + C_r^1,$$

and

$$\frac{C}{(r-\alpha)m^{r-\alpha}} \leq \frac{1}{4(r-\alpha)m^{r+1-\alpha}} + C_r^2,$$

with

$$C_r^1 := \frac{(1-\alpha)4^{\frac{r}{1-\alpha}} C^{\frac{r+1-\alpha}{1-\alpha}}}{r(r+1-\alpha)}, \quad C_r^2 := \frac{4^{r-\alpha} C^{r+1-\alpha} (r-\alpha)^{r-\alpha-1}}{(r+1-\alpha)^{r+1-\alpha}}.$$

Therefore,

$$\frac{1}{r+1-\alpha} \int_{\mathbb{T}^d} \frac{1}{m^{r-\alpha+1}} \leq 2(C_r^1 + C_r^2).$$

Thus, we get

$$\left\| \frac{1}{m} \right\|_{L^{r+1-\alpha}(\mathbb{T}^d)} \leq \left[2(r+1-\alpha)(C_r^1 + C_r^2) \right]^{\frac{1}{r+1-\alpha}} =: C_\alpha(r).$$

We can easily check that, for any $r_0 > \alpha$ there exists C_α for which

$$C_\alpha(r) \leq C_\alpha \text{ for all } r \in [r_0, \infty). \quad \square$$

Proposition 2.3. *For any $p \geq 1$ there exists a constant $C := C_p(\|b\|_\infty, \|V\|_\infty) > 0$ such that for any classical solution (u, m) of (1.1)-(1.2), we have $\|u\|_{W^{2,p}(\mathbb{T}^d)} + \|m\|_{W^{2,p}(\mathbb{T}^d)} \leq C$.*

Proof. Let (u, m) be a classical solution (u, m) to (1.1)-(1.2). In view of Lemma [AC78, Lemma 4], combined with Proposition 2.2 we conclude that for all $p \in [1, \infty)$ there exists $C = C(\|V\|_\infty, \|b\|_\infty, p)$ such that

$$\|u\|_{W^{2,p}(\mathbb{T}^d)} \leq C.$$

In light of the Sobolev embedding theorem, we get

$$\|u\|_{C^{1,\gamma}(\mathbb{T}^d)} \leq C\|u\|_{W^{2,p}(\mathbb{T}^d)} \leq C. \quad (2.3)$$

Then, multiplying (1.2) by m^p and using Young's inequality yield

$$\begin{aligned} \int_{\mathbb{T}^d} m^{p+1} dx + p \int_{\mathbb{T}^d} m^{p-1} |Dm|^2 dx &= \int_{\mathbb{T}^d} m^p dx - p \int_{\mathbb{T}^d} m^p (g + b) \cdot Dm dx \\ &\leq \left[\frac{1}{2} \int_{\mathbb{T}^d} m^{p+1} dx + C \right] + \left[\frac{p}{2} \int_{\mathbb{T}^d} m^{p-1} |Dm|^2 dx + Cp \int_{\mathbb{T}^d} (|g|^2 + |b|^2) m^{p+1} dx \right], \end{aligned}$$

where $g := Du/m^\alpha$. Noting that $|g| \leq C$ in view of (2.3) and $m^{p-1} |Dm|^2 = C_p |Dm^{(p+1)/2}|^2$, we get

$$\int_{\mathbb{T}^d} m^{p+1} dx + C_p \int_{\mathbb{T}^d} |Dm^{(p+1)/2}|^2 dx \leq C + C'_p \int_{\mathbb{T}^d} m^{p+1} dx. \quad (2.4)$$

Using Hölder's inequality we have

$$\left(\int_{\mathbb{T}^d} m^{p+1} \right)^{1/(p+1)} \leq \left(\int_{\mathbb{T}^d} m \right)^{2/(2+dp)} \left(\int_{\mathbb{T}^d} m^{2^*(p+1)/2} \right)^{\frac{dp/(2+dp)}{2^*(p+1)/2}}.$$

Moreover, using the Sobolev embedding theorem, we get

$$\begin{aligned} \int_{\mathbb{T}^d} m^{p+1} &\leq \left(\int_{\mathbb{T}^d} m^{2^*(p+1)/2} \right)^{\frac{dp/(2+dp)}{2^*/2}} \\ &\leq C \left(\int_{\mathbb{T}^d} m^{p+1} dx + \int_{\mathbb{T}^d} |Dm^{(p+1)/2}|^2 dx \right)^{dp/(2+dp)}. \end{aligned}$$

Then, using the previous estimate and the fact that $dp/(2+dp) < 1$ in the right-hand side of (2.4), we conclude that

$$\int_{\mathbb{T}^d} m^{p+1} dx + \int_{\mathbb{T}^d} |Dm^{(p+1)/2}|^2 dx \leq C. \quad (2.5)$$

Note now that if $m \in W^{1,q}(\mathbb{T}^d)$, we have

$$m - \Delta m = m^{1-\alpha} \Delta u + (1 - \alpha) m^{-\alpha} Du \cdot Dm + \operatorname{div}(mb) + 1 \in L^q(\mathbb{T}^d). \quad (2.6)$$

Thus, by standard elliptic regularity $m \in W^{2,q}(\mathbb{T}^d)$. Consequently $m \in W^{1,q^*}(\mathbb{T}^d)$. In light of (2.5) for $p = 1$ we have $m \in W^{1,2}(\mathbb{T}^d)$. Thus we obtain $m \in W^{2,2}(\mathbb{T}^d)$ and $m \in W^{1,2^*}(\mathbb{T}^d)$. By iterating this argument, we finally get $m \in W^{2,q}(\mathbb{T}^d)$ for any $q < \infty$. \square

Proposition 2.4. *For any integer $k \geq 0$ there exists a constant $C := C(\|b\|_\infty, \|V\|_\infty, k) > 0$ such that any classical solution (u, m) of (1.1)-(1.2) satisfies $\|u\|_{W^{k,\infty}(\mathbb{T}^d)} + \|m\|_{W^{k,\infty}(\mathbb{T}^d)} \leq C$.*

Proof. Note that $D(m^{-\alpha}) = m^{-(1+\alpha)}Dm \in L^p(\mathbb{T}^d)$ for large $p > 1$ in view of Propositions 2.2, 2.3. This implies $m^{-\alpha} \in W^{1,p}(\mathbb{T}^d)$. Hence, by Morrey's theorem, we have $m^{-\alpha} \in C^\gamma(\mathbb{T}^d)$ for some $\gamma \in (0, 1)$. Also note that $|Du|^2 \in C^\gamma(\mathbb{T}^d)$. Therefore, going back to the equation (1.1) we have

$$u - \Delta u = -\frac{|Du|^2}{2m^\alpha} - b \cdot Du + V \in C^\gamma(\mathbb{T}^d). \quad (2.7)$$

Then, in view of the elliptic regularity theory we get $u \in C^{2,\gamma}(\mathbb{T}^d)$. Note that the norm in $C^\gamma(\mathbb{T}^d)$ of the right hand side of (2.7) is estimated by a constant which only depends on $\|b\|_\infty, \|V\|_\infty$. Thus, $u \leq C(\|b\|_\infty, \|V\|_\infty)$. Next, going back to the equation (2.6) for m and noting that the right hand side is $C^\gamma(\mathbb{T}^d)$ now, we get $m \in C^{2,\gamma}(\mathbb{T}^d)$ with $m \leq C(\|b\|_\infty, \|V\|_\infty)$.

Once we know that $u, m \in C^{2,\gamma}(\mathbb{T}^d)$, (2.7) and (2.6) imply $u, m \in C^{3,\gamma}(\mathbb{T}^d)$. By continuing this so-called “bootstrap” argument, we get the conclusion. \square

Remark 1. The methods in this section rely strongly on the particular structure of the Hamiltonian which allows for the cancellation in (2.2). In the forthcoming paper [GM13], we address mean field games which generalize (1.1)-(1.2), namely of the form

$$\begin{cases} u - \Delta u + m^\alpha H(x, \frac{Du}{m^\alpha}) = V(x, m) & \text{in } \mathbb{T}^d \\ m - \Delta m - \operatorname{div} \left(m D_p H(x, \frac{Du}{m^\alpha}) \right) = 1, & \text{in } \mathbb{T}^d, \end{cases}$$

but for which the cancellation in (2.2) no longer holds. Such problems are natural in many applications, for instance (2.2) is not valid for the anisotropic quadratic Hamiltonian $H(x, p) = 2^{-1}|A(x)p + b(x)|^2$, where A is a strictly positive definite matrix. This is addressed in [GM13] with a different approach.

3. EXISTENCE BY CONTINUATION METHOD

In this section we prove the existence of a unique classical solution to (1.1)-(1.2) by using the continuation method. We work under the assumptions of Theorem 1.1. For $0 \leq \lambda \leq 1$ we consider the problem

$$\begin{cases} u_\lambda - \Delta u_\lambda + \frac{|Du_\lambda|^2}{2m_\lambda^\alpha} + \lambda b(x) \cdot Du_\lambda - \lambda V(x, m) - (1 - \lambda)V_0(m) = 0 & \text{in } \mathbb{T}^d, \\ m_\lambda - \Delta m_\lambda - \operatorname{div} (m_\lambda^{1-\alpha} Du_\lambda) - \lambda \operatorname{div} (bm_\lambda) = 1 & \text{in } \mathbb{T}^d, \end{cases} \quad (3.1)$$

where $V_0(m) := \arctan(m)$. Let $E^k := H^k(\mathbb{T}^d) \times H^k(\mathbb{T}^d)$ for $k \in \mathbb{N}$, and $E^0 := L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$.

For any $k_0 \in \mathbb{N}$ with $k_0 > d/2$, we define the map $F : [0, 1] \times E^{k_0+2} \rightarrow E^{k_0}$ by

$$F(\lambda, u, m) := \begin{pmatrix} u - \Delta u + \frac{|Du|^2}{2m^\alpha} + \lambda b(x) \cdot Du - \lambda V(x, m) - (1 - \lambda)V_0(m) \\ m - \Delta m - \operatorname{div} (m^{1-\alpha} Du) - \lambda \operatorname{div} (bm) - 1. \end{pmatrix}.$$

Then, we can rewrite (3.1) as

$$F(\lambda, u_\lambda, m_\lambda) = 0.$$

Note that for any $\gamma > 0$, the map F is C^∞ in the set $\{(u, m) \in E^{k_0+2}(\mathbb{T}^d), m > \gamma\}$. This is because for $k_0 > d/2$, the Sobolev space $H^{k_0}(\mathbb{T}^d)$ is an algebra. Moreover, if k_0 is large enough, then any solution (u_λ, m_λ) in E^{k_0+2} is, in fact, in E^{k+2} for all $k \in \mathbb{N}$, by the a-priori bounds in Section 2.

We define the set Λ by

$$\Lambda := \{\lambda \in [0, 1] \mid (3.1) \text{ has a classical solution } (u, m) \in E^{k_0+2}\}.$$

When $\lambda = 0$ we have an explicit solution, namely $(u_0, m_0) = (\pi/4, 1)$. Therefore, $\Lambda \neq \emptyset$. The main goal of this section is to prove

$$\Lambda = [0, 1].$$

To prove this, we show that Λ is relatively closed and open on $[0, 1]$.

The closeness of Λ is a straightforward consequence of the estimates in Section 2:

Proposition 3.1. *The set Λ is closed.*

Proof. To prove that Λ is closed, we need to check that for any sequence $\lambda_n \in \Lambda$ such that $\lambda_n \rightarrow \lambda_0$ as $n \rightarrow \infty$, we have $\lambda_0 \in \Lambda$. Fix such a sequence and corresponding solutions $(u_{\lambda_n}, m_{\lambda_n})$ to (3.1) with $\lambda = \lambda_n$. Since the a-priori bound in Proposition 2.4 is independent of $n \in \mathbb{N}$, by taking a subsequence, if necessary, we may assume that $(u_{\lambda_n}, m_{\lambda_n}) \rightarrow (u, m)$ in E^{k_0+2} . Moreover, $m_{\lambda_n}^{-1} \rightarrow m^{-1}$ in $C(\mathbb{T}^d)$. Therefore, we can take the limit in (3.1), and we get that (u, m) is solution to (3.1) with $\lambda = \lambda_0$. This implies $\lambda_0 \in \Lambda$. \square

To prove that Λ is relatively open in $[0, 1]$, we need to check that for any $\lambda_0 \in \Lambda$ there exists a neighborhood of λ_0 contained in Λ . To do so, we will use the implicit function theorem (see, for example, [Die69], chapter X). For a fixed $\lambda_0 \in \Lambda$, we consider the Fréchet derivative $\mathcal{L}_{\lambda_0} : E^{k_0+2} \rightarrow E^{k_0}$ of $(u, m) \mapsto F(\lambda_0, u, m)$ at the point $(u_{\lambda_0}, m_{\lambda_0})$, which is given by

$$\begin{aligned} & \mathcal{L}_{\lambda_0}(v, f) \\ &= \begin{pmatrix} v - \Delta v + \frac{Du_{\lambda_0} \cdot Dv}{m_{\lambda_0}^\alpha} - \frac{\alpha |Du_{\lambda_0}|^2 f}{2m_{\lambda_0}^{\alpha+1}} + \lambda_0 b \cdot Dv - (\lambda_0 D_m V + (1 - \lambda_0) D_m V_0) f \\ f - \Delta f - \operatorname{div}(m_{\lambda_0}^{1-\alpha} Dv) - (1 - \alpha) \operatorname{div}(m_{\lambda_0}^{-\alpha} f Du_{\lambda_0}) - \lambda_0 \operatorname{div}(bf) \end{pmatrix}. \end{aligned} \quad (3.2)$$

Because of the a-priori bounds for u and m in Section 2, we can extend the domain of \mathcal{L}_{λ_0} by continuity to E^{k+2} for any $k \leq k_0$. We will prove that \mathcal{L}_{λ_0} is an isomorphism from E^{k+2} to E^k for any $k \geq 0$.

Define the bilinear mapping $B_{\lambda_0}[w_1, w_2] : E^1 \rightarrow \mathbb{R}$ by

$$\begin{aligned} & B_{\lambda_0}[w_1, w_2] \\ &:= \int_{\mathbb{T}^d} \left[v_1 + \frac{Du_{\lambda_0} \cdot Dv_1}{m_{\lambda_0}^\alpha} - \frac{\alpha |Du_{\lambda_0}|^2 f_1}{2m_{\lambda_0}^{\alpha+1}} + \lambda_0 b \cdot Dv_1 - (\lambda_0 D_m V + (1 - \lambda_0) D_m V_0) f_1 \right] f_2 \\ &+ Dv_1 \cdot Df_2 - m_{\lambda_0}^{1-\alpha} Dv_1 Dv_2 \\ &+ [f_1 - (1 - \alpha) \operatorname{div}(m_{\lambda_0}^{-\alpha} f_1 Du_{\lambda_0}) - \lambda_0 \operatorname{div}(bf_1)] (-v_2) - Df_1 \cdot Dv_2 dx. \end{aligned}$$

We set $Pw := (f, -v)$ for $w = (v, f)$, and we observe that if $w_1 \in E^k$ with $k \geq 2$, then

$$B_{\lambda_0}[w_1, w_2] = \int_{\mathbb{T}^d} \mathcal{L}_{\lambda_0}(w_1) \cdot Pw_2 \, dx.$$

The boundedness of B_{λ_0} is a straightforward result of Proposition 2.4:

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$|B_{\lambda_0}[w_1, w_2]| \leq C \|w_1\|_{E^1} \|w_2\|_{E^1}.$$

for any $w_1, w_2 \in E^1$.

Thus, in view of the Riesz representation theorem for Hilbert spaces, there exists a linear mapping $A : E^1 \rightarrow E^1$ such that

$$B_{\lambda_0}[w_1, w_2] = (Aw_1, w_2)_{E^1}.$$

Lemma 3.3. *The operator A is injective.*

Proof. Let $w = (v, f)$. By Young's inequality we have

$$\begin{aligned} B_{\lambda_0}[w, w] &= \int_{\mathbb{T}^d} \frac{\alpha Du_{\lambda_0} \cdot Dv}{m_{\lambda_0}^\alpha} f - \frac{\alpha |Du_{\lambda_0}|^2}{2m_{\lambda_0}^{\alpha+1}} f^2 - (\lambda_0 D_m V + (1 - \lambda_0) D_m V_0) f^2 - m_{\lambda_0}^{1-\alpha} |Dv|^2 \, dx \\ &\leq \int_{\mathbb{T}^d} -(\lambda_0 D_m V + (1 - \lambda_0) D_m V_0) f^2 + \frac{(\alpha - 2) m_{\lambda_0}^{1-\alpha} |Dv|^2}{2} \\ &\leq -C_{\lambda_0} (\|Dv\|_{L^2(\mathbb{T}^d)}^2 + \|f\|_{L^2(\mathbb{T}^d)}^2) \end{aligned}$$

for a constant C_{λ_0} which depends on bounds for m_{λ_0} , and Du_{λ_0} , but it is strictly positive for any solution to (3.1) since $0 \leq \alpha < 1$. We have used Assumption (A1) and the strict monotonicity of V on m . This implies that if $Aw = 0$ we have $w = (\mu, 0)$, for some constant μ . Then, by computing

$$0 = (Aw, (0, \mu)) = B[(\mu, 0), (0, \mu)] = \mu^2,$$

we conclude that $\mu = 0$. □

Remark 2. Note that the injectivity of the operator A holds for all $0 \leq \alpha < 2$. However, the a-priori estimates of the previous section are only valid for $0 \leq \alpha < 1$.

Lemma 3.4. *The range $R(A)$ is closed, and $R(A) = E^1$.*

Proof. Take a Cauchy sequence z_n in the range of A , that is $z_n = Aw_n$, for some sequence $w_n = (v_n, f_n)$. We claim that w_n is a Cauchy sequence. We have

$$\begin{aligned} (z_n - z_m, w_n - w_m)_{E^1} &= (A(w_n - w_m), w_n - w_m)_{E^1} \\ &\leq -C_{\lambda_0} (\|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)}^2 + \|f_n - f_m\|_{L^2(\mathbb{T}^d)}^2). \end{aligned}$$

Note that

$$\begin{aligned}
& |(z_n - z_m, w_n - w_m)_{E^1}| \\
& \leq \|z_n - z_m\|_{E^0} \|w_n - w_m\|_{E^0} + \|D(z_n - z_m)\|_{E^0} \|D(w_n - w_m)\|_{E^0} \\
& \leq \|z_n - z_m\|_{E^0} (\|v_n - v_m\|_{L^2(\mathbb{T}^d)} + \|f_n - f_m\|_{L^2(\mathbb{T}^d)}) \\
& \quad + \|D(z_n - z_m)\|_{E^0} (\|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)} + \|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)}) \\
& \leq \varepsilon \left(\|f_n - f_m\|_{L^2(\mathbb{T}^d)}^2 + \|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)}^2 \right) + C_\varepsilon \|z_n - z_m\|_{E^1}^2 \\
& \quad + \|z_n - z_m\|_{E^1} (\|v_n - v_m\|_{L^2(\mathbb{T}^d)} + \|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)})
\end{aligned}$$

for $\varepsilon > 0$.

If we fix a suitable small ε and combine the inequalities above, then we obtain

$$\begin{aligned}
& \|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)}^2 + \|f_n - f_m\|_{L^2(\mathbb{T}^d)}^2 \\
& \leq C \|z_n - z_m\|_{E^1}^2 + C \|z_n - z_m\|_{E^1} (\|v_n - v_m\|_{L^2(\mathbb{T}^d)} + \|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)}) \\
& \leq C \|z_n - z_m\|_{E^1}^2 + C \|z_n - z_m\|_{E^1} \|w_n - w_m\|_{E^1}.
\end{aligned} \tag{3.3}$$

We have

$$B[w_n - w_m, (-f_n + f_m, v_n - v_m)] = \|w_n - w_m\|_{E^1}^2 + E_{nm},$$

where, using (3.3), E_{nm} satisfies

$$\begin{aligned}
|E_{nm}| & \leq C \|v_n - v_m\|_{L^2(\mathbb{T}^d)} \|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)} + C \|f_n - f_m\|_{L^2(\mathbb{T}^d)} \|v_n - v_m\|_{L^2(\mathbb{T}^d)} \\
& \quad + C \|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)} \|D(v_n - v_m)\|_{L^2(\mathbb{T}^d)} \\
& \quad + C \|f_n - f_m\|_{L^2(\mathbb{T}^d)} \|D(f_n - f_m)\|_{L^2(\mathbb{T}^d)} \\
& \leq C \|w_n - w_m\|_{E^1} (\|z_n - z_m\|_{E^1}^2 + \|z_n - z_m\|_{E^1} \|w_n - w_m\|_{E^1})^{1/2}.
\end{aligned} \tag{3.4}$$

On the other hand, by Lemma 3.2 we have

$$B[w_n - w_m, (-f_n + f_m, v_n - v_m)] \leq C \|z_n - z_m\|_{E^1} \|w_n - w_m\|_{E^1}. \tag{3.5}$$

Combining (3.4) and (3.5) we deduce

$$\begin{aligned}
& \|w_n - w_m\|_{E^1}^2 \\
& \leq C \|z_n - z_m\|_{E^1} \|w_n - w_m\|_{E^1} \\
& \quad + C \|w_n - w_m\|_{E^1} (\|z_n - z_m\|_{E^1}^2 + \|z_n - z_m\|_{E^1} \|w_n - w_m\|_{E^1})^{1/2}.
\end{aligned}$$

By using Young's inequality we conclude

$$\|w_n - w_m\|_{E^1}^2 \leq C \|z_n - z_m\|_{E^1}^2.$$

From this we get convergence in E^1 .

Finally, we prove that $R(A) = E^1$. Suppose that $R(A) \neq E^1$. Since $R(A)$ is closed, there would exist $z \in R(A)^\perp$ with $z \neq 0$ such that $B_{\lambda_0}[z, z] = 0$. The argument in the proof of Lemma 3.3 implies $z = 0$ which is a contradiction. \square

Lemma 3.5. *The operator $\mathcal{L}_{\lambda_0} : E^{k+2} \rightarrow E^k$ is an isomorphism for all $k \in \mathbb{N}$ with $k \geq 2$.*

Proof. Since \mathcal{L}_{λ_0} is injective, it suffices to prove that it is surjective. To do so, fix $w_0 \in E^k$ with $w_0 = (v_0, f_0)$. We claim there exists a solution $w_1 \in E^{k+2}$ to $\mathcal{L}_{\lambda_0} w_1 = w_0$.

Consider the bounded linear functional $w \mapsto (w_0, w)_{E^0}$ in E^1 . By the Riesz representation theorem, there exists $\tilde{w} \in E^1$ such that $(w_0, w)_{E^0} = (\tilde{w}, w)_{E^1}$ for any $w \in E^1$. In light of Lemmas 3.3 and 3.4, there exists the inverse of A . We define $w_1 := A^{-1}\tilde{w}$, and write $w_1 = (v, f)$. Set

$$\begin{pmatrix} g_1[v, f] \\ g_2[v, f] \end{pmatrix} := \begin{pmatrix} \frac{Du_{\lambda_0} \cdot Dv}{m_{\lambda_0}^\alpha} - \frac{\alpha |Du_{\lambda_0}|^2 f}{2m_{\lambda_0}^{\alpha+1}} + \lambda_0 b \cdot Dv - (\lambda_0 D_m V + (1 - \lambda_0) D_m V_0) f \\ -\operatorname{div}(m_{\lambda_0}^{1-\alpha} Dv) - (1 - \alpha) \operatorname{div}(m_{\lambda_0}^{-\alpha} f Du_{\lambda_0}) - \lambda_0 \operatorname{div}(bf) \end{pmatrix}.$$

Then, the identity

$$(Aw_1, w)_{E^1} = (\tilde{w}, w)_{E^1} = (w_0, w)_{E^0}$$

for any $w \in E^1$, means that v is a weak $H^1(\mathbb{T}^d)$ solution to

$$v - \Delta v = g_1[v, f] + v_0,$$

and that $f \in H^1(\mathbb{T}^d)$ is also a weak solution to

$$f - \Delta f = g_2[v, f] + f_0.$$

Observe that if $v, f \in H^{j+1}(\mathbb{T}^d)$ then $g_1, g_2 \in H^j(\mathbb{T}^d)$. Additionally, elliptic regularity yields, from $g_i[v, f] \in H^j(\mathbb{T}^d)$, that $v, f \in H^{j+2}(\mathbb{T}^d)$. Since we have $v, f \in H^1(\mathbb{T}^d)$, we conclude by induction that $v, f \in H^{j+2}(\mathbb{T}^d)$, for all $j \leq k$. \square

A straightforward result of Lemma 3.5 and the implicit function theorem in Banach space is

Proposition 3.6. *The set Λ is relatively open in $[0, 1]$.*

We finally address the existence of solutions to (1.1), (1.2), and complete the proof of Theorem 1.1.

Proof of Theorem 1.1. If V is strictly increasing on m , the existence of a classical solution to (1.1), (1.2) is a straightforward result of Proposition 3.6. Uniqueness of solution is discussed in the next appendix, Proposition A.1.

If we only assume V to be nondecreasing on m , existence can be obtained by using a perturbation argument similar to the one in (3.1). More precisely, we add a small perturbation $\varepsilon \arctan(m)$ to V so that we make the potential term strictly monotone. This problem admits a unique classical solution $(u_\varepsilon, m_\varepsilon)$. Because the a-priori bounds in the previous section do not depend on the strict monotonicity of V , $(u_\varepsilon, m_\varepsilon)$ satisfy uniform bounds in any Sobolev space. Thus, by compactness, we can extract a convergent subsequence to a limit (u, m) which solves (1.1)-(1.2). \square

Remark 3. In this paper, we focus on the case where V is bounded on m , since our main concern is a lower bound on m . In principle, unbounded potentials V can be studied by adapting the techniques in [LL07a, CLLP12, GPSM12, GPSM14], for instance.

APPENDIX A. UNIQUENESS

Uniqueness of solutions of (1.1)-(1.2) is well understood (see [LL06b, Gue14] for a related problem). However, to make this paper self-contained, we give a proof based on Lions ideas in [Lio11].

Proposition A.1. *The system (1.1)-(1.2) admits at most one classical solution (u, m) .*

Proof. Let (u_0, m_0) and (u_1, m_1) be classical solutions to (1.1)-(1.2). Subtract (1.1) for (u_1, m_1) from (1.1) for (u_0, m_0) , and (1.2) for (u_1, m_1) from (1.2) for (u_0, m_0) , respectively, and then

$$u_0 - u_1 = \Delta(u_0 - u_1) + \frac{|Du_1|^2}{2m_1^\alpha} - \frac{|Du_0|^2}{2m_0^\alpha} + b \cdot D(u_1 - u_0) + V(x, m_0) - V(x, m_1), \quad (\text{A.1})$$

$$m_0 - m_1 = \Delta(m_0 - m_1) + \operatorname{div}(m_0^{1-\alpha} Du_0) - \operatorname{div}(m_1^{1-\alpha} Du_1) + \operatorname{div}(b(m_0 - m_1)). \quad (\text{A.2})$$

Subtract (A.2) multiplied by $u_0 - u_1$ from (A.1) multiplied by $m_0 - m_1$, and then

$$\begin{aligned} & \int_{\mathbb{T}^d} \left(\frac{|Du_1|^2}{2m_1^\alpha} - \frac{|Du_0|^2}{2m_0^\alpha} \right) (m_0 - m_1) dx + \int_{\mathbb{T}^d} (m_0^{1-\alpha} Du_0 - m_1^{1-\alpha} Du_1) \cdot D(u_0 - u_1) dx \\ &= \int_{\mathbb{T}^d} (V(x, m_1) - V(x, m_0))(m_0 - m_1) dx. \end{aligned} \quad (\text{A.3})$$

We prove that the left-hand side of (A.3) is non-negative if $\alpha \in [0, 2]$ following the technique in [Gue14]. Set $u_\theta := u_0 + \theta(u_1 - u_0)$ and $m_\theta := m_0 + \theta(m_1 - m_0)$ for $\theta \in [0, 1]$. Define

$$\begin{aligned} I(\theta) := & \left[- \int_{\mathbb{T}^d} \left(\frac{|Du_\theta|^2}{2m_\theta^\alpha} - \frac{|Du_0|^2}{2m_0^\alpha} \right) (m_1 - m_0) \right. \\ & \left. + \int_{\mathbb{T}^d} (m_\theta^{1-\alpha} Du_\theta - m_0^{1-\alpha} Du_0) \cdot D(u_1 - u_0) dx \right]. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} I(\theta) &= -\alpha \int_{\mathbb{T}^d} \frac{Du_\theta \cdot D(u_1 - u_0)(m_1 - m_0)}{m_\theta^\alpha} dx \\ &\quad + \frac{\alpha}{2} \int_{\mathbb{T}^d} \frac{|Du_\theta|^2 (m_1 - m_0)^2}{m_\theta^{1+\alpha}} dx + \int_{\mathbb{T}^d} m_\theta^{1-\alpha} |D(u_1 - u_0)|^2 dx \\ &\geq \left(1 - \frac{\alpha}{2}\right) \int_{\mathbb{T}^d} m_\theta^{1-\alpha} |D(u_1 - u_0)|^2 dx \geq 0 \end{aligned}$$

for $\alpha \in [0, 2]$. Noting that $I(0) = 0$, we conclude that $I(1) \geq 0$ which is the claim.

Thus, by (A.3) and the assumption that V is strictly increasing on m , we get the conclusion. \square

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